A Geometric Photon Mass-Gap Conjecture: Spectral Gaps for Photonic Operators

Peter De Ceuster

October 14, 2025

Abstract

We formulate a precise *Geometric Photon Mass-Gap Conjecture* asserting that Laplace-type operators modelling photonic dynamics on compact manifolds with photonic coupling admit a strictly positive spectral gap above zero under explicit geometric and bundle-theoretic hypotheses. We define a class of Laplace-type photonic operators L_{ph} acting on sections of vector bundles with photonic connection and potential terms, state the conjecture with an explicit lower bound depending on curvature, flux, injectivity radius and volume, and prove a special-case lemma establishing positivity of the first eigenvalue under a coercive potential hypothesis. Heuristics based on heat-kernel asymptotics, Weitzenböck identities and Cheeger-type inequalities are given to motivate the form of the bound. Finally we outline a numerical verification plan using finite-element eigenvalue solvers for our geometries and discuss possible experimental signatures of a photonic mass gap.

Contents

1 Introduction **1**

2 Operator setup **2**

3 Main conjecture **2**

4 Spectral lemmas and special-case proof **3**
- 4.1 Weitzenböck and Bochner techniques (heuristic) 3
- 4.2 Cheeger-type and Poincaré lower bounds (heuristic) 3

5 Numerical verification plan **3**
- 5.1 Discretization 4
- 5.2 Eigenvalue computation and parameter scans 4
- 5.3 Validation and sensitivity 4

6 Discussion and physical consequences **4**

7 Conclusion **4**

1 Introduction

A mass gap for a field operator is the statement that the spectrum of the associated kinetic operator has a strictly positive lower bound above zero. In electromagnetic contexts the absence of a mass gap for the Maxwell operator reflects the massless photon; however, in media with geometric coupling (e.g. topological photonic crystals, background gerbe fluxes, or constrained bundles) an effective spectral gap can appear. Motivated by geometric field theory and spectral geometry, we propose a conjecture guaranteeing such a gap for a class of Laplace-type photonic operators on compact Riemannian manifolds with bundle-valued photonic couplings.

The conjecture provides an explicit lower bound in terms of geometric invariants (curvature, photonic flux norms, injectivity radius, volume) and bundle data (connection curvature, potential lower bounds). We prove a supporting lemma in the elementary coercive-potential case and outline analytic and numerical evidence for the general statement.

2 Operator setup

Let (M, g) be a closed (compact, without boundary) Riemannian manifold of dimension $d \geq 2$. Let $\pi : E \to M$ be a Hermitian vector bundle of rank r with a unitary connection ∇^E. Let $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$ denote its curvature.

Definition 2.1 (Photonic coupling data). *A* photonic coupling *is a triple* (∇^E, B, V) *where*

- ∇^E *is a unitary connection on* E*,*
- $B \in \Omega^1(M, \mathrm{End}(E))$ *is a real-valued (skew-hermitian) 1-form potential encoding photonic coupling/flux interactions (e.g. effective vector potential from medium),*
- $V \in C^\infty(M, \mathrm{Herm}(E))$ *is a fiberwise self-adjoint potential (possibly arising from dielectric contrast or nonlinear mean-field effects).*

Definition 2.2 (Photonic Laplace-type operator). *Given photonic coupling* (∇^E, B, V) *define the* photonic operator L_{ph} *acting on smooth sections* $u \in C^\infty(M; E)$ *by the Laplace-type formula*

$$L_{\mathrm{ph}} u \;=\; -(\nabla^{E,B})^* \nabla^{E,B} u \;+\; V u, \tag{2.1}$$

where $\nabla^{E,B} := \nabla^E + B$ *is the twisted connection (summing the 1-form* B *interpreted as an endomorphism-valued* 1*-form), and* $(\nabla^{E,B})^*$ *is the* L^2*-adjoint with respect to the Riemannian metric and bundle metric. Equivalently* L_{ph} *is a formally self-adjoint second-order elliptic operator of Laplace-type.*

We equip L_{ph} with domain $H^2(M; E)$ as an unbounded self-adjoint operator on $L^2(M; E)$.

3 Main conjecture

We now state the Geometric Photon Mass-Gap Conjecture in a precise, quantitative form.

Conjecture 3.1 (Geometric Photon Mass-Gap Conjecture). *Let* (M, g) *be a closed Riemannian manifold with injectivity radius* $\mathrm{inj}(M)$ *and Ricci curvature bounded below by* $\mathrm{Ric}_g \geq -(d-1)\kappa^2 g$ *for some* $\kappa \geq 0$*. Let* $E \to M$ *be a Hermitian vector bundle with photonic coupling* (∇^E, B, V) *such that*

1. *the curvature of the twisted connection obeys the pointwise bound* $\|F_{\nabla^{E,B}}(x)\|_{\mathrm{op}} \leq \Lambda_F$ *for all* $x \in M$*,*
2. *the potential has a positive uniform lower bound* $V(x) \succeq v_0 \, \mathrm{Id}_E$ *for some* $v_0 \in \mathbb{R}$*,*
3. *the* L^∞*-norm of the photonic 1-form satisfies* $\|B\|_{L^\infty} \leq B_0$*.*

Then there exists a constant

$$C = C\big(d, \mathrm{Vol}(M), \mathrm{inj}(M), \kappa, \Lambda_F, B_0\big) \tag{3.1}$$

explicitly computable from the listed geometric invariants such that the bottom of the spectrum of L_{ph} *satisfies*

$$\lambda_1\big(L_{\mathrm{ph}}\big) \;\geq\; \max\{\, v_0,\; C \,\}. \tag{3.2}$$

In particular, if $v_0 > 0$ *or if the geometric/flux constant* $C > 0$*, then* L_{ph} *admits a strictly positive spectral gap above* 0 *bounded below by the right-hand side of* (3.2)*.*

Remark 3.2. *The constant* C *is intended to capture coercivity arising from curvature, flux and topological constraints even when* $v_0 \leq 0$*. When* V *is strictly positive the bound* (3.2) *is immediate; the nontrivial content is that geometry and photonic flux alone can force a positive gap.*

4 Spectral lemmas and special-case proof

We prove a supporting lemma in the coercive-potential case; this shows Conjecture 3.1 holds with $C = v_0$ and indicates the structure of more subtle estimates.

Lemma 4.1 (Coercive potential lemma)**.** *Let L_{ph} be as in* (2.1) *and assume $V(x) \succeq v_0 \mathrm{Id}_E$ for some $v_0 > 0$. Then the spectrum of L_{ph} is contained in $[v_0, \infty)$ and in particular $\lambda_1(L_{\mathrm{ph}}) \geq v_0$.*

Proof sketch. For $u \in C^\infty(M; E)$ integrate by parts and use self-adjointness:

$$\langle u, L_{\mathrm{ph}} u \rangle_{L^2} = \|\nabla^{E,B} u\|_{L^2}^2 + \langle u, Vu \rangle_{L^2}.$$

By the pointwise bound on V we obtain

$$\langle u, L_{\mathrm{ph}} u \rangle_{L^2} \geq \|\nabla^{E,B} u\|_{L^2}^2 + v_0 \|u\|_{L^2}^2 \geq v_0 \|u\|_{L^2}^2.$$

Therefore the Rayleigh quotient satisfies

$$\frac{\langle u, L_{\mathrm{ph}} u \rangle}{\langle u, u \rangle} \geq v_0,$$

and taking the infimum over the domain yields $\lambda_1 \geq v_0$. The conclusion follows by the spectral theorem for self-adjoint operators. □

Remark 4.2. *The lemma is elementary but important: positive potentials trivially produce a gap. The challenge is to quantify positive lower bounds arising purely from geometry and photonic flux when V may not be positive.*

4.1 Weitzenböck and Bochner techniques (heuristic)

When L_{ph} connects to a connection Laplacian one can use a Weitzenböck (Bochner) identity of the form

$$(\nabla^{E,B})^* \nabla^{E,B} = \nabla^* \nabla + \mathcal{R}(x), \tag{4.1}$$

where $\mathcal{R}(x) \in \mathrm{End}(E)$ is a curvature/endormorphism term depending on Ricci curvature and bundle curvature $F_{\nabla^{E,B}}$. If a pointwise lower bound $\mathcal{R}(x) \succeq r_0 \mathrm{Id}_E$ holds then coercivity as in Lemma 4.1 is recovered with v_0 replaced by r_0. Thus geometric positivity of $\mathcal{R}$ (e.g. positive Ricci curvature combined with curvature pinching) yields a mass gap.

4.2 Cheeger-type and Poincaré lower bounds (heuristic)

Cheeger inequalities relate isoperimetric constants to the first nonzero scalar Laplacian eigenvalue. For bundle-valued operators with photonic coupling one can sometimes reduce to scalar estimates via positivity-preserving assumptions or diamagnetic inequalities; these yield lower bounds of the type

$$\lambda_1(L_{\mathrm{ph}}) \geq \frac{h^2}{4},$$

where h is an appropriate isoperimetric constant depending on (M, g) and coupling. Heat-kernel short-time asymptotics can be used to read off leading-order geometric contributions to spectral density; combining short-time control with functional inequalities leads to candidate expressions for C in (3.1).

5 Numerical verification plan

To aid future work, we outline a computational approach to test Conjecture 3.1.

5.1 Discretization

1. Choose sample manifolds M amenable to discretization (flat tori, spheres, or discretized curved surfaces embedded in $\mathbb{R}^3$).
2. Implement finite element (FEM) spaces for vector-valued sections of E (e.g. Nédélec elements for 1-forms if modelling electromagnetic vector fields).
3. Discretize the twisted connection $\nabla^{E,B}$ via covariant finite elements, and form the discrete stiffness matrix approximating $(\nabla^{E,B})^*\nabla^{E,B}$ plus mass matrix for V.

5.2 Eigenvalue computation and parameter scans

Use sparse eigensolvers (ARPACK, SLEPc) to compute the lowest eigenvalues of the discrete operator for ranges of parameters (B_0, Λ_F, v_0) and geometric deformations. Extrapolate discretization convergence by mesh refinement and estimate numerical lower bounds for λ_1. In theory, from there onward one could advance toward Validation.

5.3 Validation and sensitivity

Compare numerical λ_1 against the analytic bounds in special regimes (e.g. coercive V) and measure sensitivity of the gap to photonic flux B and curvature by controlled experiments (e.g. adding localized flux tubes, varying curvature on a family of deformed spheres).

6 Discussion and physical consequences

A verified positive lower bound $\lambda_1 > 0$ for L_{ph} implies absence of arbitrarily low-energy photonic eigenmodes in the given geometric and bundle setting: physically, this corresponds to an effective photonic mass gap preventing propagation below a cutoff frequency. Such gaps have implications for localization, robustness of topological modes, and design of photonic insulators. The conjecture situates these phenomena in a rigorous spectral-geometric framework. Further attention and efforts will be needed to expand our new idea.

7 Conclusion

We proposed a Geometric Photon Mass-Gap Conjecture (Conjecture 3.1) giving an explicit lower bound for the first eigenvalue of Laplace-type photonic operators in terms of geometric and bundle invariants. A simple coercive lemma (Lemma 4.1) supports the conjecture in the presence of a positive potential; spectral geometry techniques and numerical strategies were sketched to attack the general case. Establishing Conjecture 3.1 would link geometric/topological photonic design to rigorous analytic spectral bounds and provide a mathematical underpinning for photonic mass-gap phenomena.